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QUALITATIVE ANALYSIS OF FORMS OF MOTION IN THE PROBLEM OF THE MOTION OF AN ARTIFICIAL SATELLITE IN THE NORMAL GRAVITATIONAL FIELD OF THE EARTH

Ye. P. Aksenov, Ye. A. Grebenikov, V. G. Demin

Introduction

In the theory of motion of artificial satellites whose orbits lie outside the earth's atmosphere, the basic perturbing factor is the deviation of the earth's gravitational field from a central field. In an earlier paper [1] the present authors showed that the potential of the earth's gravitational attraction is fairly well approximated, with accuracy to second-degree terms with respect to flattening of the earth, by the potential function of the problem of two fixed centers of equal mass situated at some imaginary distance from each other. The expansion of this potential function, which we term the normal gravitational field of the earth, is expressed by the formula /173

$$U_0 = \frac{fM}{r}\left\{1 + \sum_{k=1}^{\infty} (-1)^k \left(\frac{c}{r}\right)^{2k} P_{2k}\left(\frac{z}{r}\right)\right\}, \qquad (1)$$

where $P_{2k}\left(\frac{z}{r}\right)$ is a Legendre polynomial of order 2k; $r = \sqrt{x^2 + y^2 + z^2}$; x, y, z are rectangular geocentric equatorial coordinates of the satellite; f is the gravitational constant;

[redacted]

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M is the earth's mass; $c = 0.0331406\ R_0$; R_0 is the equatorial radius of the earth (the value of c is determined on the basis of I. D. Zhongolovich's results [2]).

Satellite motion is studied in generalized elliptical coordinates μ, λ, w, related to the rectangular coordinates by the formulas

$$\left.\begin{aligned} x &= c\sqrt{(1+\lambda^2)(1-\mu^2)}\ \sin w, \\ y &= c\sqrt{(1+\lambda^2)(1-\mu^2)}\ \cos w, \\ z &= -c\lambda\mu. \end{aligned}\right\} \qquad (2)$$

The elliptical coordinates μ, λ, w are determined from the following first integrals of the equations of motion:

$$\left(\frac{d\mu}{d\tau}\right)^2 = 2h\mu^4 + 2(c_2 - h)\mu^2 - (2c_2 + c_1^2), \qquad (3)$$

$$\left(\frac{d\lambda}{d\tau}\right)^2 = -2h\lambda^4 - \frac{2fM}{c^3}\lambda^3 + 2(c_2 - h)\lambda^2 - \frac{2fM}{c^3}\lambda + (2c_2 + c_1^2), \qquad (4)$$

$$\frac{dw}{d\tau} = \frac{c_1(\mu^2 + \lambda^2)}{(1 - \mu^2)\,(1 + \lambda^2)} \,. \qquad (5)$$

The symbols h, c_1, c_2 denote arbitrary constants of integration, with the constant h related to the magnitude of the total mechanical energy h_1 of the satellite in the following manner: /174

$$hc^2 = - h_1. \qquad (6)$$

In formulas (3)-(5), τ is an independent regularizing variable related to the time t by the differential equality

$$\frac{d\tau}{dt} = \frac{1}{\mu^2 + \lambda^2} \,. \qquad (7)$$

From the conversion formulas (2), it is clear that real values of the rectangular coordinate correspond to

$$|\,\mu\,| \leq 1. \qquad (8)$$

Note also that the conversion formulas (2) may be used to derive the equations

$$\frac{x^2 + y^2}{c^2(1 + \lambda^2)} + \frac{z^2}{c^2\lambda^2} = 1, \tag{9}$$

$$\frac{x^2 + y^2}{c^2(1 - \mu^2)} - \frac{z^2}{c^2\mu^2} = 1, \tag{10}$$

$$\frac{x}{\sin w} - \frac{y}{\cos w} = 0. \tag{11}$$

Equation (9) demonstrates that if the differential equations of motion allow a solution λ = const, then the moving point will be lie consistently on an ellipsoid of rotation. Similarly, if μ = const is a solution of the equations of motion, then the moving point will lie on a hyperboloid of revolution of one sheet (10), and, finally, if there exists a solution w = const, then the motion of the point will take place in the meridional plane passing through the Oz axis.

All forms of motion in the problem discussed may be broken down into three categories depending on the value of h . Motions corresponding to $h > 0$ will be termed elliptical motions. Motions corresponding to $h < 0$ will belong to the class of hyperbolic motions. The class of parabolic motions will embrace all trajectories for which $h = 0$.

This classification of motions is accounted for by the argument that when c is put equal to zero in the equations of motion, these equations revert to the differential equations of the two-body problem, and then the motions belonging to the first class will take place on Keplerian ellipses, the trajectories belonging to the second class will be hyperbolas, and the trajectories belonging to the third class will be Keplerian parabolas.

Among the trajectories defined by the differential equations of the problem will be found some which lie entirely within the interior of the earth. These trajectories are of no value in practice, and we shall therefore ignore them in the discussion. Motions occurring on trajectories which pass even partially outside of the earth's surface will be termed real motions. Among these motions will be found some which take place on restricted trajectories which lie entirely outside the earth's surface. These motions will be termed satellite motions. Ballistic trajectories will be the term applied to those restricted trajectories lying partially inside the earth.

1. INVESTIGATION OF THE ELLIPTICAL COORDINATE μ

The general solution of the differential equation defining the elliptical coordinate μ depends on the roots of the polynomial /175

$$f(\mu) = 2h\mu^4 + 2\,(c_2 - h)\,\mu^2 - (2c_2 + c_1^2)\,. \quad (12)$$

The roots of the polynomial $f(\mu)$ are found from the expression

$$\mu = \pm\sqrt{\frac{h - c_2 \pm \sqrt{(h + c_2)^2 + 2hc_1^2}}{2h}} . \qquad (13)$$

Let the roots of $f(\mu)$ be $\pm\mu_1$, $\pm\mu_2$. The discriminant $\Delta = (h + c_2)^2 + 2hc_1^2$ for the class of elliptical motions $(h > 0)$ will always take on a nonnegative value. From this, we infer that of four roots belonging to the polynomial $f(\mu)$, at least two (which we shall term $\pm\mu_1$ to be specific) will be real numbers, whereupon

$$|\mu_1| \geq 1. \qquad (14)$$

Depending on the values of the other two roots $\pm\mu_2$, we may encounter the following cases illustrated in Fig. 1: a) μ_2 is a complex quantity; b) μ_2 is a real number satisfying the constraint $|\mu_2| > 1$; c) μ_2 is a real number and $|\mu_2| < 1$; d) $\mu_2 = 0$; e) μ_2 is a real number equal to 1, $|\mu_2| = 1$; f) $|\mu_1| = |\mu_2| = 1$.

From formula (12), we infer that the domain of real μ values is defined by the inequality

$$f(\mu) \geq 0. \qquad (15)$$

In case a), the polynomial $f(\mu)$ satisfies condition (15) at $|\mu| \geq |\mu_1| \geq 1$, but at these values of the variable μ, the rectangular coordinates x and y will not be real. We may therefore neglect case a). Case b) need not be considered either, since the inequality $|\mu_2| > 1$ is equivalent to the inequality $2hc_1^2 < 0$, which cannot occur in the case of elliptical motions. For case d), it follows that $z = 0$, i.e. we obtain equatorial orbits. Case e) is realized when $c_1 = 0$. But under that condition we infer from eq. (5) that $w = w_0$, i.e., the case of polar orbits. Cases d) and e) will be discussed in detail in a separate article. Case f) may occur at $c_1 = 0$, $h = -c_2$, and in that case polar orbits will also occur.

For case c), we infer from the condition $0 < |\mu| < 1$ that

$$2c_2 + c_1^2 < 0. \tag{16}$$

The inequality (16) aids considerably in simplifying the search for roots of the polynomial dependent on the elliptic coordinate λ.

The spatial motions of a particle are consequently possible only in case c), with the exception of some motions in the plane.

The function $f(\mu)$ may be factored into

$$f(\mu) = 2h\,(\mu^2 - \mu_1^2)\,(\mu^2 - \mu_2^2). \tag{17}$$

Then, from equation (3), we obtain:

$$\int \frac{d\mu}{\sqrt{(\mu^2 - \mu_1^2)\ (\mu^2 - \mu_2^2)}} = \sqrt{2h}(\tau + c_3)\ . \qquad (18)$$

As a result of the inversion of the elliptic integral (18), we obtain the elliptic coordinate μ as a function of the regularizing time τ in the form /176

$$\mu = \mu_2 \operatorname{sn} \sigma\ (\tau - \tau_0), \qquad (19)$$

where

$$\sigma = \mu_1 \sqrt{2h}\ , \qquad (20)$$

τ_0 is an integration constant. The modulus of the elliptic function μ is

$$k = \frac{\mu_2}{\mu_1}\ . \qquad (21)$$

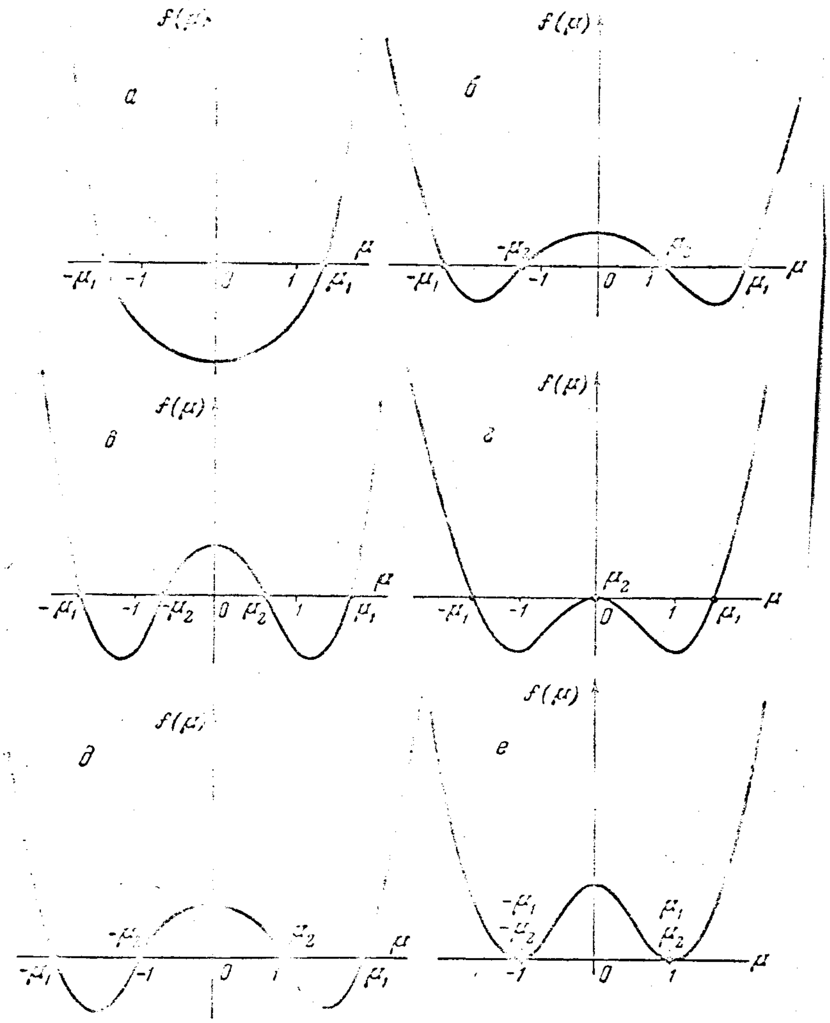

Fig. 1. Possible forms of the plot of $f(\mu)$, corresponding to various values of μ_2:

a. complex quantity; b. real number $(|\mu_2| > 1)$;
c. real number $(|\mu|_2 < 1)$; d. $\mu_2 = 0$;
e. real number $(|\mu_2| = 1)$; f. real number $(|\mu_1| = |\mu_2| = 1)$.

2. STUDY OF THE QUADRATURE DEFINING THE ELLIPTIC COORDINATE λ, CRITERION FOR THE EXISTENCE OF REAL MOTIONS

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Integration of equation (4) yields

$$\int \frac{d\lambda}{\sqrt{\psi(\lambda)}} = \tau + c_4 , \tag{22}$$

where the polynomial $\psi(\lambda)$ is expressed by the formula

$$\psi(\lambda) = -2h\lambda^4 - \frac{2fM}{c^3}\lambda^3 + 2(c_2 - h)\lambda^2 - \frac{2fM}{c^3}\lambda + (2c_2 + c_1^2). \tag{23}$$

We designate the roots of the polynomial $\psi(\lambda)$ as λ_1, λ_2, λ_3, λ_4.

All coefficients of the polynomial $\psi(\lambda)$ are negative, considering the inequality (16), so that its real roots can be only negative. Depending on the values of the roots of the polynomial $\psi(\lambda)$, we may distinguish the cases:

a) $\psi(\lambda)$ has imaginary roots

b) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 < 0.$

c) $\lambda_1 = \lambda_2 = \lambda_3 < \lambda_4 < 0.$

d) $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 0.$

e) $\lambda_1 < \lambda_2 = \lambda_3 = \lambda_4 < 0.$

f) $\lambda_1 = \lambda_2 < \lambda_3 = \lambda_4 < 0.$

g) $\lambda_1 = \lambda_2 < \lambda_3 < \lambda_4 < 0.$

h) $\lambda_1 < \lambda_2 = \lambda_3 < \lambda_4 < 0.$

i) $\lambda_1 < \lambda_2 < \lambda_3 = \lambda_4 < 0.$

j) $\lambda_1 = \lambda_2 < 0$; roots λ_3 and λ_4 imaginary.

k) $\lambda_1 < \lambda_2 < 0$; roots λ_3 and λ_4 imaginary.

In our subsequent discussion, we shall demonstrate that no real motions occur in some of the cases listed. The existence of real motions requires the fulfillment of the following constraint inferred from formula (9):

$$c^2 (1 + \lambda^2) > R^2_{pol} \tag{24}$$

where R_{pol} is the polar radius of the earth, or

$$|\lambda| > 30. \tag{25}$$

For convenience, we consider, instead of the polynomial $\psi(\lambda)$, a newly introduced polynomial $\varphi(\lambda)$ having the same roots as $\psi(\lambda)$:

$$\varphi(\lambda) = \lambda^4 + a\lambda^3 + b\lambda^2 + a\lambda + d. \tag{26}$$

The coefficients a, b, and d are obtained from the formulas

$$a = \frac{fM}{c^3 h}, \qquad b = 1 - \frac{c_2}{h}, \qquad d = -\frac{2c_2 + c_1^2}{2h} \tag{27}$$

These coefficients for spatial motions belonging to the elliptical class are positive, since

$$b > d + 1. \tag{28}$$

Note that no real motions occur in case a, since $\psi(\lambda) < 0$.

3. TREATMENT OF THE CASE b ($\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$) /178

The polynomial $\varphi(\lambda)$ is plotted graphically in this case, in Fig. 2.

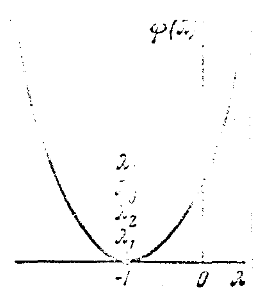

Fig. 2. Plot of the polynomial $\varphi(\lambda)$ in the case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$.

Note that if real motions were to occur in this case, they would have to take place on the ellipsoid

$$\frac{x^2 + y^2}{c^2(1+\lambda_1^2)} + \frac{z^2}{c^2\lambda_1^2} = 1. \tag{29}$$

Bearing in mind the fact that the coefficients of λ and λ^3 in the polynomial $\varphi(\lambda)$ are equal, and using Vieta's theorem, we may obtain

$$\lambda_1 = \lambda_1^3 . \tag{30}$$

Hence, λ_1 is either equal to 0 or to $\pm$ 1. But none of these proposed roots will satisfy the criterion for the existence of real motions, and we conclude that there are then no real motions, in this case.

4. TREATMENT OF THE CASE c ($\lambda_1 = \lambda_2 = \lambda_3 < \lambda_4 < 0$)

The graphical behavior of the polynomial $\varphi(\lambda)$ in this case is seen in Fig. 3. For this case, Vieta's theorem yields the following equations

$$\left.\begin{aligned} 3\lambda_1 + \lambda_4 &= -a, \\ 3\lambda_1^2 + 3\lambda_1\lambda_4 &= b, \\ \lambda_1^3 + 3\lambda_1^2\lambda_4 &= -a, \\ \lambda_1^3\lambda_4 &= d. \end{aligned}\right\} \qquad (31)$$

From the first and third of equations (31), we find that

$$\frac{\lambda_1}{\lambda_4} = \frac{1 - 3\lambda_1^2}{\lambda_1^2 - 3}\,. \qquad (32)$$

Taking into account the fact that all the roots of $\varphi(\lambda)$ are negative, we shall have

$$\frac{1 - 3\lambda_1^2}{\lambda_1^2 - 3} > 0. \qquad (33)$$

This last inequality is satisfied at

$$-\sqrt{3} < \lambda_1 < -\frac{1}{\sqrt{3}} \qquad (34)$$

Since the region where motion is possible is defined by the inequality $\lambda_1 < \lambda < \lambda_4$, the elliptic coordinate will not exceed $\sqrt{3}$ in absolute value. This last constraint stands in contradiction to the criterion for the existence of real motions. We then infer that no real motions occur in this case.

5. THE CASE d $(\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 0)$ /179

We shall now prove that if all the roots of $\varphi(\lambda)$ are real and unequal, then satellite and ballistic motions exist. Moreover, we shall prove that two roots of the four are included in the open interval $(-1, 0)$. The function $\varphi(\lambda)$ is plotted in Fig. 4.

The Sturm system [3] for the polynomial $\varphi(\lambda)$ consists of the functions

$$\left.\begin{aligned} \varphi(\lambda) &= \lambda^4 + a\lambda^3 + b\lambda^2 + a\lambda + d, \\ \varphi_1(\lambda) &= 4\lambda^3 + 3a\lambda^2 + 2b\lambda + a, \\ \varphi_2(\lambda) &= a_2\lambda^2 + b_2\lambda + d_2, \\ \varphi_3(\lambda) &= a_3\lambda + b_3, \\ \varphi_4(\lambda) &= a_4. \end{aligned}\right\} \quad (35)$$

The coefficients of functions (35) are expressed by the formulas

$$
\left.\begin{aligned}
a_2 &= \frac{3a^2 - 8b}{16}, \quad b_2 = \frac{ab - 6a}{8}, \quad d_2 = \frac{a^2 - 16d}{16}, \\
a_3 &= \frac{-3a^4 + a^2b^2 + 14a^2b - 18a^2 - 4b^3 - 6a^2d - 16bd}{8a_2^2}, \\
b_3 &= \frac{a^3b + 3a^3 - 9a^3d + 32abd - 48ad - 4ab^2}{16a_2^2} \\
a_4 &= \frac{a_3b_2b_3 - a_3^2d_2 - a_2b_3^2}{a_3^2} .
\end{aligned}\right\} \qquad (36)
$$

We infer from Sturm's theorem [3] that all the roots of the polynomial $\varphi(\lambda)$ are real and unequal on the open interval $(-\infty, +\infty)$, if the inequalities

$$
\left.\begin{aligned}
a_2 &> 0, \\
a_2 &> 0, \\
a_4 &> 0.
\end{aligned}\right\} \qquad (37)
$$

all hold simultaneously. Although we know all the roots to be negative, we nevertheless consider the interval $(-\infty, +\infty)$ in order investigate the least number of inequalities compris- /180 ing the system (37). To solve the system (37), we examine the following functions in many variables:

$$\left.\begin{aligned}
z_1 &= 3x^2 - 8y,\\
z_2 &= -3x^4 + x^2y^2 + 14x^2y - 18x^2 - 4y^3 - d\,(6x^2 - 16y),\\
z_3 &= -36x^{10} + 9x^8y^2 - 84x^6y^3 + 354x^8y + 256x^4y^4 -\\
&\quad - 243x^8 - 1168x^6y^2 + 1152x^6y + 1344x^4y^3 - 256x^2y^5 -\\
&\quad - 1728x^4y^2 + d\cdot f\,(x, y, d),
\end{aligned}\right\} \tag{38}$$

where $f\,(x, y, d)$ is a polynomial of not higher than ninth degree in x and y. The inequalities (37) are equivalent to the inequalities

$$\left.\begin{aligned}
z_1 &> 0,\\
z_2 &> 0,\\
z_3 &> 0.
\end{aligned}\right\} \tag{39}$$

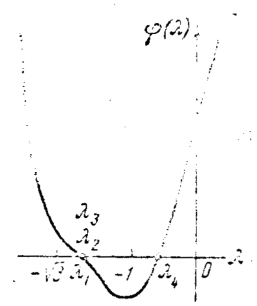

Fig. 3. Slope of polynomial $\varphi(\lambda)$ in the case $\lambda_1=\lambda_2=\lambda_3<\lambda_4<0$.

To solve the inequalities (39), we resort to polar coordinates. The functions z_1, z_2, z_3 in polar coordinates assume the form

$$\left.\begin{aligned} z_1 &= 3r^2 \cos^2 \varphi - 8r \sin \varphi, \\ z_2 &= a_0(\varphi) r^4 + a_1(\varphi) r^3 + a_2(\varphi, d) r^2 + a_3(\varphi, d) r, \\ z_3 &= b_0(\varphi) r^{10} + \sum_{k=2}^{9} b_k(\varphi, d) r^k. \end{aligned}\right\} \quad (40)$$

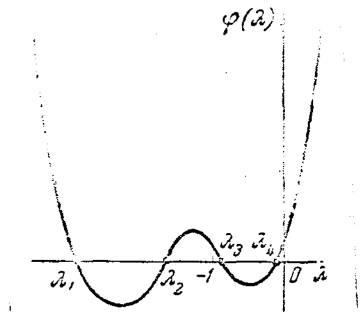

Fig. 4. Slope of polynomial $\varphi(\lambda)$ in the case $\lambda_1 < \lambda_2 < < \lambda_3 < \lambda_4 < 0$.

The coefficients $a_0\ (\varphi)$ and $b_0\ (\varphi)$ are expressed by the formulas

$$a_0(\varphi) = \cos^2 \varphi\ (\sin^2 \varphi - 3 \cos^2 \varphi),\ b_0(\varphi) = 9 \cos^8 \varphi(\sin^2 \varphi - 4 \cos^2 \varphi). \tag{41}$$

At sufficiently large values of r, the inequalities (39) are fulfilled when

$$\left.\begin{aligned} a_0\ (\varphi) &> 0, \\ b_0\ (\varphi) &> 0. \end{aligned}\right\} \tag{42}$$

For positive values of x and y ($x = a$, $y = b$), the constraints (42) are fulfilled when

$$2 < \tan \varphi, \quad \text{arc tan } 2 < \varphi < \frac{\pi}{2} \ . \tag{43}$$

Consequently, for large values of the coefficients a and b, the polynomial $\varphi(\lambda)$ has four real and unequal roots and, in accord with the upper-bound theorem for the absolute roots of a polynomial [3], the roots may be fairly large, generally speaking.

We shall now show that the roots λ_3 and λ_4 are included in the open interval $(-1, 0)$. For this, we apply the Budan-Fourier theorem [3] to the interval $(-1,0)$. The polynomial $\varphi(\lambda)$ and its derivatives to the 4-th order have the form

$$\left.\begin{aligned}
\varphi(\lambda) &= \lambda^4 + a\lambda^3 + b\lambda^2 + a\lambda + d,\\
\varphi'(\lambda) &= 4\lambda^3 + 3a\lambda^2 + 2b\lambda + a,\\
\varphi''(\lambda) &= 12\lambda^2 + 6a\lambda + 2b,\\
\varphi'''(\lambda) &= 24\lambda + 6a,\\
\varphi^{IV}(\lambda) &= 24.
\end{aligned}\right\} \tag{44}$$

When $\lambda = 0$, $\varphi(0) > 0$, $\varphi'(0) > 0$, $\varphi'(0) > 0$, $\varphi'''(0) > 0$, /181

$\varphi^{IV}(0) > 0$. If $\lambda = -1$, then it is mandatory that $\varphi(-1) > 0$, since $\varphi(-1) < 0$ would be equivalent to the condition $\tan \varphi < 2$, but this stands in contradiction to the constraints (43). Note also that, for real motions, $\varphi'''(-1) > 0$, i.e., $a > 4$. If we assume that $\varphi'''(-1) < 0$, then this will mean that $a < 4$, $b < 6$, $d < 5$. But in this case, the roots of the polynomial would also be less than 6 in absolute value. Consequently, the variation in the signs of the functions (44) at $\lambda = -1$ would occur, in the case of real motions, in one of the four variants:

	I	II	III	IV
$\varphi(-1)$	+	+	+	+
$\varphi'(-1)$	+	-	-	+
$\varphi''(-1)$	-	-	+	+
$\varphi'''(-1)$	+	+	+	+
$\varphi^{IV}(-1)$	+	+	+	+

The values of the functions (44) at $\lambda = -1$ will be found from the formulas

$$\varphi(-1) = 1 - 2a + b + d,$$

$$\varphi'(-1) = -4 + 4a - 2b,$$

$$\left.\begin{aligned} \varphi''(-1) &= 12 - 6a + 2b, \\ \varphi'''(-1) &= -24 + 6a, \\ \varphi^{IV}(-1) &= 24. \end{aligned}\right\} \quad (45)$$

The corresponding inequalities for the cases I, II, and III are not contradictory. For the case IV, we find that the first and second derivatives will be positive when $a < 4$, but, as indicated earlier, this case constitutes unreal motions.

The inversion of the elliptic integral (22) for the case of real and unequal roots yields, for the variable λ, the expression

$$\lambda = \frac{A + B\ \mathrm{sn}^2\sigma_1\ (\tau - \tau_1)}{C + D\ \mathrm{sn}^2\sigma_1\ (\tau - \tau_1)}\ , \quad (46)$$

where

$$\left.\begin{aligned} A &= \lambda_2\ (\lambda_3 - \lambda_1), & B &= \lambda_3\ (\lambda_1 - \lambda_2)\ , \\ C &= \lambda_3 - \lambda_1\ , & D &= \lambda_1 - \lambda_2\ , \end{aligned}\right\} \quad (47)$$

$$\sigma_1 = \frac{\sqrt{2h(\lambda_4 - \lambda_2)\ (\lambda_3 - \lambda_4)}}{2}\ . \tag{48}$$

The modulus of the elliptic sine, k_1 is expressed by the formula

$$k_1 = \sqrt{\frac{(\lambda_4 - \lambda_3)\ (\lambda_2 - \lambda_1)}{(\lambda_4 - \lambda_2)\ (\lambda_3 - \lambda_1)}}\ . \tag{49}$$

The coordinate w is defined from the quadrature

$$w = c_5 + c_1 \int \frac{(\mu^2 + \lambda^2)\ d\tau}{(1 - \mu^2)\ (1 + \lambda^2)} \tag{50}$$

or /182

$$w = c_5 + c_1 J_1 - c_1 J_2\ , \tag{51}$$

where

$$J_1 = \int \frac{d\tau}{1 - \mu^2}, \tag{52}$$

$$J_2 = \int \frac{d\tau}{1 + \lambda^2}. \tag{53}$$

The integral J_1 for all the cases treated in the article manifests the same form:

$$J_1 = \frac{1}{c}\Pi(\varphi, n, k), \tag{54}$$

where $\Pi(\varphi, n, k)$ is an elliptic integral of the third kind:

$$\varphi = \text{am}\,[\sigma\,(\tau - \tau_0)], \tag{55}$$

$$n = -\mu_2^2. \tag{56}$$

The integral J_2 [after substituting $x = \text{sn}\,\sigma_1\,(\tau - \tau_1)$]

is expressed in the form

$$J_2 = \frac{\tau}{1 + \lambda_3^2} + \int \frac{(a_0 x^2 + a_1)\, dx}{(b_0 x^4 + b_1 x^2 + b_2) \sqrt{(1 - x^2)\,(1 - k_1^2 x^2)}} . \tag{57}$$

The coefficients in (57) are expressed in terms of the coefficients A, B, C, D by the formulas

$$\left.\begin{aligned} a_0 &= \frac{2BD\,(BC - AD)}{\sigma_1 (B^2 + D^2)}, \quad a_1 \frac{B^2C^2 - A^2D^2}{\sigma_1 (B^2 + D^2)}, \\ b_0 &= B^2 + D^2, \quad b_1 = 2\,(AB + CD), \\ b_2 &= A^2 + C^2 . \end{aligned}\right\} \tag{58}$$

The second term in formula (57) appears in the form of the sum of two elliptic integrals of the third kind with complex conjugate parameters. These integrals may be expressed in terms of elliptic integrals of the third kind with real parameters [4]. Here, we present only the form of these expressions, without going into detail as to the values of all the coefficients, which are expressed in a rather cumbersome manner in terms of the roots of the polynomial $\varphi(\lambda)$. Note that it is not feasible to make use of the formulas mentioned in practical calculations. In practice, it would be better to resort to expansions of elliptic functions and elliptic integrals, bearing in mind the fact that the moduli of the elliptic functions and integrals are extremely small quantities in the case of satellite motions.

Taking the Hoüle transformation [4] into account, the integral J_2 may be stated in the form

$$J_2 = \frac{\tau}{1 + \lambda_3^2} + \iota_1 L_1 + \iota_2 L_2, \tag{59}$$

where the functions L_1 and L_2 have the form

$$L_i = \int \frac{dx}{1+hx^2} - \frac{F(\mathrm{am}[\sigma_1(\tau - \tau_1)])}{g_i} - Q_i \Pi(\mathrm{am}[\sigma_1(\tau - \tau_1)];\ n_i),$$

$$(i = 1,\ 2). \tag{60}$$

The coefficients ι_1, ι_2, g_i, n_i, Q_i are related to (58) in a cumbersome fashion. /183

Accordingly, we derive a formula of the form:

$$w = c_5 - \frac{c_1 \tau}{1 + \lambda_3^2} + \frac{c_1}{\sigma} \Pi (\varphi, n, k) - c_1 \iota_1 L_1 - c_1 \iota_2 L_2. \tag{61}$$

for the coordinate w.

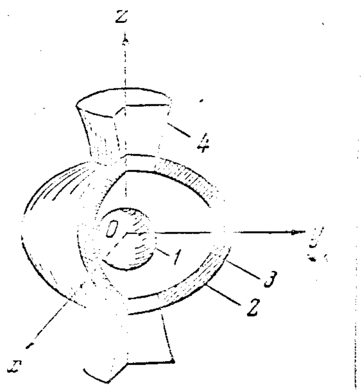

Fig. 5. Region of possible orbit positions in the case $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 0$:
1. earth; 2. outer ellipsoid; 3. inner ellipsoid; 4. hyperboloid; region containing orbits latched.

The relationship between the true time and the regularizing time is found from a formula of the type

$$t = c_6 + (\mu_1^2 + \lambda_3^2)\,\tau - \frac{\mu_1^2}{\sigma}\,E\,(\varphi, k) + \overline{\iota_1 L_1} + \overline{\iota_2 L_2}, \qquad (62)$$

where $E\,(\varphi, k)$ is an elliptic integral of the second kind, and the functions $\overline{L}_1$ and $\overline{L}_2$ are expressed by formulas similar to (60). Note that formula (62) should not be used in practice, since the expansion of the integrand functions into series with subsequent integration will substantially reduce in the computational labor.

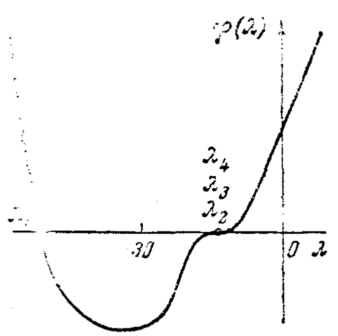

Fig. 6. Slope of polynomial $\varphi(\lambda)$ in the case $\lambda_1 < \lambda_2 = \lambda_3 = \lambda_4 < 0$.

In the case d, the trajectory lies between two confocal ellipsoids $\lambda = \lambda_1$ and $\lambda = \lambda_2$ and outside the hyperboloid $\dfrac{x^2 + y^2}{c^2(1-\mu_2^2)} - \dfrac{z^2}{c^2\mu_2^2} = 1$ (Fig. 5).

If $|\lambda_2| < 30$, we will have ballistic trajectories, i.e., in this case one of the ellipsoids will lie inside the earth, and the region where the trajectories may be accomodated will occupy the volume outside of the hyperboloid from the earth's surface to the ellipsoid (just as in case e, below).

The trajectories will be tangent to the ellipsoids in all cases (or to the ellipsoid, when ballistic trajectories are involved), and to the hyperboloid by virtue of the continuity of the derivatives with respect to τ of the coordinates x, y, and z.

6. THE CASE e ($\lambda_1 < \lambda_2 = \lambda_3 = \lambda_4 < 0$) /184

The polynomial $\varphi(\lambda)$ for this case is plotted in Fig. 6.

The equations expressing Vieta's theorem for case e are given by the formulas

$$\left.\begin{aligned} &\lambda_1 + 3\lambda_2 = -a, \\ &3\lambda_1\lambda_2 + 3\lambda_2^2 = b, \\ &3\lambda_1\lambda_2^2 + \lambda_2^3 = -a, \\ &\lambda_1\lambda_2^3 = d. \end{aligned}\right\} \qquad (63)$$

From the first and third equations, we find that the existence of real motions ($\lambda_1 < -30$) requires that the inequality

$$\frac{\lambda_2 (\lambda_2^2 - 3)}{1 - 3\lambda_2^2} < -30. \qquad (64)$$

be satisfied.

All solutions of the inequality (64) lie within the intervals $-0.59 < \lambda_2 < -0.58$, $0.56 < \lambda_2 < 0.58$, $90.5 < \lambda_2$. Taking into account the fact that the value of λ_2 must be negative, we may write that $-0.59 < \lambda_2 < -0.58$. At these values, $b > d + 1$.

Accordingly, in case e only ballistic trajectories are possible. The coordinates μ, λ, and w are expressed as follows:

$$\mu = \mu_2 \operatorname{sn}\sigma (\tau - \tau_0),$$

$$\lambda = \lambda_2 + \frac{\lambda_1 - \lambda_2}{1 + A_1 (\tau - \tau_1)^2},$$

$$w = c_5 - c_1\tau + \frac{c_1}{\sigma} \Pi (\varphi, n, k) -$$

$$- \frac{2c_1}{(\lambda_2-\lambda_1)\sqrt{2h}} \left\{ \frac{M_1}{2} \ln \frac{(u-p)^2+q^2}{(u+p)^2+q^2} + \frac{M_1 p+N_1}{2} \text{arc tan} \frac{2qu}{p^2+q^2-u^2} \right\} , \tag{65}$$

where

$$u = \sqrt{A_1}\,(\tau - \tau_1), \quad A_1 = \frac{h}{2}(\lambda_2 - \lambda_1)^2, \tag{66}$$

$$\left. \begin{aligned} M_1 &= \frac{2\lambda_2(\lambda_2 - \lambda_1)\sqrt{1 + \lambda_1^2 + \lambda_2^2 + \lambda_1^2\lambda_2^2} - (\lambda_2^2 - \lambda_1^2)\,(1 + \lambda_2^2)}{4p\,(1 + \lambda_2^2)\sqrt{1 + \lambda_1^2 + \lambda_2^2 + \lambda_1^2\lambda_2^2}} , \\ N_1 &= \frac{\lambda_2^2 - \lambda_1^2}{\sqrt{1 + \lambda_1^2 + \lambda_2^2 + \lambda_1^2\lambda_2^2}} , \end{aligned} \right\} \tag{67}$$

$$p = \sqrt{\frac{\sqrt{1 + \lambda_1^2 + \lambda_2^2 \lambda{}_1^2\lambda_2^2} - 1 - \lambda_1\lambda_2}{2\,(1 + \lambda_2^2)}} ,$$

$$q = \sqrt{\frac{\sqrt{1 + \lambda_1^2 + \lambda^2{}_2 + \lambda_1^2\lambda_2^2} + 1 + \lambda_1\lambda_2}{2\ (1 + \lambda_2^2)}}\ . \qquad (68)$$

The relationship linking the time t and the variable τ has the form /185

$$t = c_6 + (\mu_1^2 + \lambda_2^2)\tau + \frac{(\lambda_2 - \lambda_1)^2\ (\tau - \tau_1)}{2\ (1 + u^2)} -$$

$$- \frac{\lambda_1 + 3\lambda_2}{\sqrt{2h}} \text{ arc tan } u - \frac{\mu_1^2}{\sigma} E \left\{ \text{am}[\sigma(\tau - \tau_0)];\ k \right\}. \qquad (69)$$

An idea of the arrangement of the trajectories in this case may be gained from Fig. 5; the trajectories fill the space between the earth's surface and the ellipsoid $\lambda = \lambda_1$.

7. THE CASE f (POLYNOMIAL $\varphi(\lambda)$ HAS TWO PAIRS OF MULTIPLE ROOTS)

The function $\varphi(\lambda)$ for this case is plotted in Fig. 7.

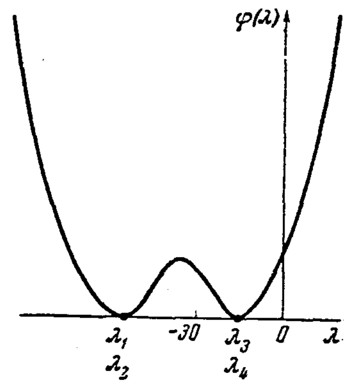

Fig. 7. Slope of polynomial $\varphi(\lambda)$ in the case $\lambda_1 = \lambda_2 < < \lambda_3 = \lambda_4 < 0$.

We shall prove that real motions are possible in this case only on the surface of the ellipsoid $\lambda = \lambda_1$, with the ellipsoid $\lambda = \lambda_3$ completely inside the earth. In fact, the relationship linking the roots and coefficients of polynomial $\varphi(\lambda)$ here displays the form

$$2\lambda_1 + 2\lambda_3 = - a,$$

$$\lambda_1^2 + 4\lambda_1\lambda_3 + \lambda_3^2 = b,$$

$$\left.\begin{aligned} 2\lambda_1^2\lambda_3 + 2\lambda_1\lambda_3^2 &= -a, \\ \lambda_1^2\lambda_3^2 &= d. \end{aligned}\right\} \tag{70}$$

From the first and third equations in (70), we may arrive at

$$\lambda_1 + \lambda_3 = \lambda_1\lambda_3(\lambda_1 + \lambda_3) \tag{71}$$

or

$$\lambda_1\lambda_3 = 1. \tag{72}$$

Consequently, when $\lambda_1 \leq -30$, then $-\frac{1}{30} \leq \lambda_3 < 0$, i.e., the ellipsoid corresponding to $\lambda = \lambda_3$ will be completely inside the earth, if we take into account the criterion for the existence of real motions (25). From the last equation in (70) and from (72), we see that $d = 1$. Coefficient b will be larger than 4, as we realize from the second equation in (70). Note that $\lambda = \lambda_1$ satisfies the equations of motion. The arguments offered lead us to the conclusion that real motions are possible on the ellipsoid.

The formulas for the relationships linking the elliptic coordinates and the time to τ in this case take on the particularly straightforward form:

$$\left.\begin{aligned}
\mu &= \mu_2 \operatorname{sn} \sigma (\tau - \tau_0),\\
\lambda &= \lambda_1,\\
w &= c_5 - \frac{c_1}{1 + \lambda_1^2}\tau + \frac{c_1}{\sigma}\prod(\varphi, n, k),\\
t &= c_6 + (\mu_1^2 + \lambda_1^2)\,\tau + \frac{\mu_1^2}{\sigma} E\,(\varphi, k).
\end{aligned}\right\} \quad (73)$$

The rectangular coorsinates of the point x, y, z are expressed by the formulas /186

$$\left.\begin{aligned}
x &= a_1 \sqrt{1 - \mu_2^2 \operatorname{sn}^2 [\sigma(\tau - \tau_0)]}\,\sin w,\\
y &= a_1 \sqrt{1 - \mu_2^2 \operatorname{sn}^2 [\sigma(\tau - \tau_0)]}\cos w,\\
z &= - b_1 \mu_2 \operatorname{sn} [\sigma(\tau - \tau_0)],
\end{aligned}\right\} \quad (74)$$

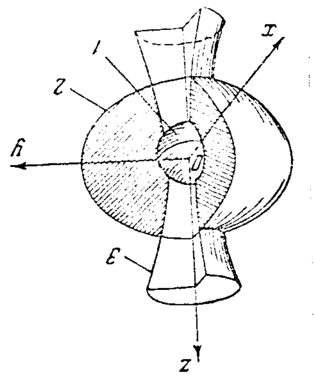

Fig. 8. Region where orbit motions are possible in the case when $\varphi(\lambda)$ has two pairs of equal roots:

1. earth; 2. ellipsoid;

3. hyperboloid; orbits lie in hatched region.

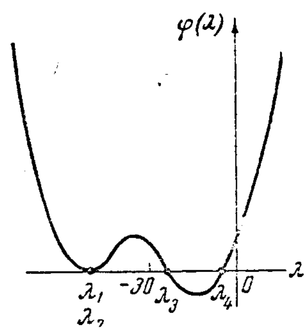

Fig. 9. Slope of polynomial $\varphi(\lambda)$ in the case $\lambda_1 = \lambda_2 < \lambda_3 < \lambda_4 < 0$.

where a_1, b_1 are the semiaxes of an ellipsoid on which motion occurs:

$$\left.\begin{aligned} a_1 &= c\sqrt{1 + \lambda_1^2}, \\ b_1 &= c\lambda_1. \end{aligned}\right\} \qquad (75)$$

The trajectories, generally speaking, fill the elliptic belt (Fig. 8) everywhere, with the intersection of the ellipsoid $\frac{x^2 + y^2}{a_1^2} + \frac{z^2}{b_1^2} = 1$ and the hyperboloid $\frac{x^2 + y^2}{c^2(1 - \mu_2^2)} - \frac{z^2}{c^2\mu_2^2} = 1$.

obtained as a result. In this case, the trajectories will be tangent to their lines of intersection.

8. THE CASE g $(\lambda_1 = \lambda_2 < \lambda_3 < \lambda_4 < 0)$

The function $\varphi(\lambda)$ corresponding to this case is plotted in Fig. 9.

If all the roots of the polynomial $\varphi(\lambda)$ are real, with only one of them (the least) a double root, then the relationships expressing Vieta's theorem will be given by the formulas

$$\left.\begin{aligned} 2\lambda_1 + \lambda_3 + \lambda_4 &= -a, \\ \lambda_1^2 + 2\lambda_1 (\lambda_3 + \lambda_4) + \lambda_3\lambda_4 &= b, \\ \lambda_1^2 (\lambda_3 + \lambda_4) + 2\lambda_1\lambda_3\lambda_4 &= -a, \\ \lambda_1^2\lambda_3\lambda_4 &= d. \end{aligned}\right\} \qquad (76)$$

We see from equations (76) that

$$\lambda_3 + \lambda_4 = \frac{2\lambda_1}{1 - \lambda_1^2} (\lambda_3\lambda_4 - 1). \qquad (77)$$

Now consider (77) in greater detail. The criterion for the existence of real motions (25) is fulfilled when /187

$$\lambda_1 < -30. \qquad (78)$$

Under this condition $\dfrac{2\lambda_1}{1 - \lambda_1^2} > 0$, and

$$\max_{\lambda_1 < -30} \left| \frac{2\lambda_1}{1 - \lambda_1^2} \right| \approx \frac{1}{15} . \qquad (79)$$

Taking into account formulas (78) and (79), we proceed from (77) to

$$0 < \lambda_3 \lambda_4 < 1. \qquad (80)$$

But this constraint in turn implies that

$$| \lambda_3 + \lambda_4 | < 1. \qquad (81)$$

The formulas advanced show that when the ellipsoid $\lambda = \lambda_1$ has dimensions greater than those of the earth, then the values $\lambda = \lambda_3$ and $\lambda = \lambda_4$ correspond to ellipsoids whose semiaxes are less than unity and which are completely inside the earth.

Inequality (28) $b > d + 1$ for that case may be stated as follows:

$$\frac{1 - \lambda_1^2}{2\lambda_1} > \frac{2\lambda_1}{1 - \lambda_1^2}. \qquad (82)$$

Inequality (82) is always fulfilled at values $\lambda_1 < -30$.

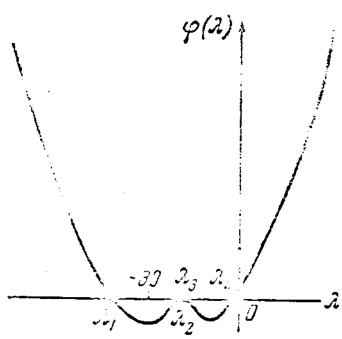

Fig. 10. Slope of polynomial $\varphi(\lambda)$ in the case $\lambda_1 < \lambda_2 = \lambda_3 < \lambda_4 < 0$.

Hence it can be concluded that real motions are possible in the ellipsoidal belt in this instance. The formulas for the elliptic and the rectangular coordinates have the same form as the formulas in the preceding section, except for the differences in some of the expressions for constant coefficients.

9. THE CASE h ($\lambda_1 < \lambda_2 = \lambda_3 < \lambda_4 < 0$)

The polynomial $\varphi(\lambda)$ for this case is plotted in Fig. 10.
The relationships between the roots and coefficients of the polynomial $\varphi(\lambda)$ (Vieta theorem) are given in this case by the equations

$$\left.\begin{aligned} &\lambda_1 + 2\lambda_2 + \lambda_4 = - a, \\ &2\lambda_1\lambda_2 + 2\lambda_2\lambda_4 + \lambda_2^2 + \lambda_1\lambda_4 = b, \\ &\lambda_1\lambda_2^2 + 2\lambda_1\lambda_2\lambda_4 + \lambda_2^2\lambda_4 = - a, \\ &\lambda_1\lambda_2^2\lambda_4 = d. \end{aligned}\right\} \qquad (83)$$

From (83), we may derive the relationship

$$\lambda_1 + \lambda_4 = \frac{2\lambda_2}{1 - \lambda_2^2} (\lambda_1 \lambda_4) - 1), \qquad (84)$$

and, taking into account the constraint $b > d + 1$, we obtain

$$(\lambda_2^2 - 1) (\lambda_1 \lambda_4 - 1) < 2\lambda_2(\lambda_1 + \lambda_4). \qquad (85)$$

We shall show that only ballistic trajectories are possible in case h, i.e., we shall show that the ellipsoids corresponding to the values $\lambda = \lambda_2$ and $\lambda = \lambda_4$ are completely inside the earth when $\lambda_1 < -30$. /188

Assume $\lambda_4 < -30$. Then the roots λ_1 and λ_2 will be, a fortiori, less than -30. But, under these conditions, $\frac{2\lambda_2}{1-\lambda_2^2} > 0$ and $\lambda_1 \lambda_4 -1 > 0$. Taking these inequalities into account, we find that the right-hand member of (84) is a positive quantity, and, because the roots λ_1 and λ_4 are negative, the left-hand member of that equation will be negative. Consequently, if we assume that all the ellipsoids envelop the earth, equation (84) will not be satisfied.

Assume further $\lambda_2 < -30$, but $-30 < \lambda_4 < 0$. Then we shall have

$$0 < \frac{2\lambda_2}{1 - \lambda_2^2} < \frac{1}{15} \quad . \tag{86}$$

For (84) and (85) to be fulfilled, necessarily $\lambda_1\lambda_4 - 1 < 0$. This in turn implies that $\left| \frac{2\lambda_2}{1 - \lambda_2^2} (\lambda_1\lambda_4 - 1) \right| < \frac{1}{15}$. But, in that case, $|\lambda_1 + \lambda_4| < \frac{1}{15}$, and this contradicts the inequalities $\lambda_1 < -30$ and $\lambda_2 < -30$. Hence, we may infer that the ellipsoids $\lambda = \lambda_2$ and $\lambda = \lambda_4$ lie completely inside the earth.

We shall now demonstrate that a set of roots λ_1, λ_2, and λ_4 may be so chosen as to satisfy (84) and (85) as well as the criterion for the existence of real motions. From (84), we

$$\lambda_1 = \frac{\lambda_4 (\lambda_2^2 - 1) - 2\lambda_2}{1 - \lambda_2^2 - 2\lambda_2\lambda_4}. \tag{87}$$

This equation, in the case of near-zero negative values of λ_4, and at values $\lambda_2 = -1 - \epsilon$, where ϵ is a sufficiently small positive quantity, will yield values less than -30 for λ_1. Under these conditions, (85), which are equivalent to the inequality b > d + 1, will also be valid.

In sum, we may draw the conclusion that, if all the roots of the polynomial $\varphi(\lambda)$ are real, and the middle root of these has a multiplicity of two, then there exist only ballistic trajectories among the real motions. Motion will occur in the space between the surface of the earth and the ellipsoid $\lambda = \lambda_1$.

The formulas for the elliptic coordinates μ, λ, w and t as functions of the regularizing variable τ will be of the form

$$
\left.
\begin{aligned}
&\mu = \mu_2 \operatorname{sn} \sigma (\tau - \tau_0), \\
&\lambda = \frac{\lambda_4(\lambda_2 - \lambda_1)(1 - e^u)^2 + \lambda_1(\lambda_4 - \lambda_2)(1 + e^u)^2}{(\lambda_2 - \lambda_1)(1 - e^u)^2 + (\lambda_4 - \lambda_2)(1 + e^u)^2}, \\
&w = c_4 + \frac{Ac_1}{a_4}\tau + \frac{c_1}{\sigma}\prod(\varphi, n, k) - \\
&- \frac{c_1}{a_4\sqrt{2h(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_2)}}\Big\{\frac{M_1}{2}\ln[(e^u - m_1)^2 + n_1^2] + \\
&+ \frac{M_2}{2}\ln[(e^u - m_2)^2 + n_2^2] + \frac{M_1 m_1 + N_1}{n_1}\arctan\frac{e^u - m_1}{n_1} + \\
&\qquad + \frac{M_2 m_2 + N_2}{n_2}\arctan\frac{e^u - m_2}{n_2}\Big\},
\end{aligned}
\right\}
\tag{88}
$$

where

$$u = \sqrt{2h\ (\lambda_4 - \lambda_2)(\lambda_2 - \lambda_1)}\ (\tau - \tau_1) \tag{89}$$

and m_1, m_2 and n_1, n_2 are, respectively, the real and imaginary /189
parts of the complex conjugate roots of the equation

$$a_4x^4 + a_3x^3 + a_2x^2 + a_3x + a_4 = 0, \tag{90}$$

$$\left.\begin{aligned} a_2 &= 6 - 4a^2 + 6a^4 + 6\lambda_4^2\ 4a^2\lambda_1\lambda_4 + 6a^4\lambda_1^2\ , \\ a_3 &= -\ 4 + 4a^4 - 4\lambda_4^2 + 4a^4\lambda_1^2\ , \\ a_4 &= (1 + a^2)^2 + (\lambda_4 + a^2\lambda_1)^2\ . \end{aligned}\right\} \tag{91}$$

The quantities A, M_1, M_2, N_1, N_2 are defined by the equations

$$\left.\begin{aligned}
&A + M_1 + M_2 = (1 + a^2)^2, \\
&-2\,(m_1 + m_2)\,A - 2\,m_2 M_1 - 2m_1 M_2 + N_1 + N_2 = -4 + 4a^4, \\
&(m_1^2 + m_2^2 + 4m_1 m_2 + n_1^2 + n_2^2)\,A + (m_2^2 + n_2^2)\,M_1 + \\
&\quad + (m_1^2 + n_1^2)\,M_2 - 2m_2 N_1 - 2m_1 N_2 = 6 - 4a^2 + 6a^4, \\
&[-2m_1\,(m_2^2 + n_2^2) - 2m_2\,(m_1^2 + n_1^2)]\,A + (m_2^2 + n_2^2)\,N_1 + \\
&\quad + (m_1^2 + n_1^2)\,N_2 = -4 + 4a^4, \\
&(m_1^2 + n_1^2)\,(m_2^2 + n_2^2)\,A = (1 + a^2)^2,
\end{aligned}\right\} \tag{92}$$

$$t = c_6 + (\mu_1^2 + \lambda_2^2)\tau - \frac{\mu_1^2}{\sigma}\,E\,(\varphi, k) + P_1 \cdot \frac{se^u - 1}{e^{2u} - 2se^u + 1} + Q_1 \text{ arc tan } \frac{e^u - s}{r}\,, \tag{93}$$

where

$$s = \frac{2\lambda_2 - \lambda_1 - \lambda_4}{\lambda_4 - \lambda_1}\,,$$

$$r = \frac{2\sqrt{(\lambda_2 - \lambda_1)\ (\lambda_4 - \lambda_2)}}{\lambda_4 - \lambda_1}, \tag{94}$$

$$\left.\begin{aligned} P_1 &= \frac{1}{8\sqrt{2h\ (\lambda_2 - \lambda_1)^{3/2}(\lambda_4 - \lambda_2)^{3/2}}}[- 16\lambda_2^2\lambda_4^2 + \\ &+ 16\lambda_1^2\lambda_2^2 + 16\lambda_2^4 + 16\lambda_1^2\lambda_4^2 - 32\lambda_1\lambda_2^3 - 32\lambda_2^3\lambda_4 + \\ &+ 64\lambda_1\lambda_2^2\lambda_4 - 32\lambda_1\lambda_2\lambda_4^2 - 32\lambda_1^2\lambda_2\lambda_4\ ], \\ Q_1 &= \frac{2(2\lambda_2 - \lambda_1 - \lambda_4)}{\sqrt{(\lambda_2 - \lambda_1)\ (\lambda_4 - \lambda_2)}}\ P_1 \quad \frac{4\lambda_2}{\sqrt{2h}} . \end{aligned}\right\} \tag{95}$$

10. THE CASE i ($\lambda_1 < \lambda_2 < \lambda_3 = \lambda_4 < 0$)

The polynomial $\varphi(\lambda)$ for the case i is plotted in Fig. 11.
Vieta's theorem for the roots of the polynomial $\varphi(\lambda)$ is expressed by means of the equations

$$\left.\begin{aligned}
&\lambda_1 + \lambda_2 + 2\lambda_3 = -a, \\
&\lambda_1\lambda_2 + 2\lambda_3(\lambda_1 + \lambda_2) + \lambda_3^2 = b, \\
&2\lambda_1\lambda_2\lambda_3 + \lambda_3^2(\lambda_1 + \lambda_2) = -a, \\
&\lambda_1\lambda_2\lambda_3^2 = d.
\end{aligned}\right\} \qquad (96)$$

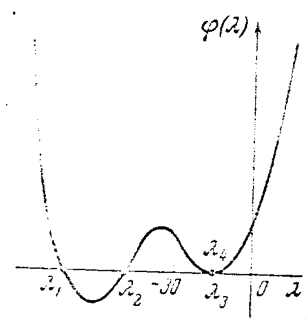

Fig. 11. Slope of the polynomial $\varphi(\lambda)$ in the case $\lambda_1 < \lambda_2 < \lambda_3 = \lambda_4 < 0$.

In a manner similar to (84) and (85), we here exhibit the formulas /190

$$\left.\begin{aligned} \lambda_1 + \lambda_2 &= \frac{2\lambda_3}{1 - \lambda_3^2}\,(\lambda_1 \lambda_s - 1), \\ \lambda_1 + \lambda_2 &< \frac{\lambda_3^2 - 1}{2\lambda_3}\,(\lambda_1 \lambda_2 - 1). \end{aligned}\right\} \quad (97)$$

It is readily shown that the ellipsoid corresponding to the value of the multiple root $\lambda = \lambda_3$ lies inside the earth. In fact, the assumption that $\lambda_3 < -30$ implies the inequality $0 < \lambda_1 \lambda_2 < 1$. This inequality cannot be valid since $\lambda_1 < \lambda_2 < \lambda_3$ and hence $\lambda_1 < -30$ and $\lambda_2 < -30$. On the other hand, it is clear from (97) that if the multiple root λ_3 is chosen on the open interval $(-1, 0)$, then these relationships will hold, at least that the root λ_1 may be made less than -30.

The motion of the point will take place between the ellipsoids

$$\frac{x^2 + y^2}{c^2\,(1 + \lambda_1^2)} + \frac{z^2}{c^2 \lambda_1^2} = 1,$$

$$\frac{x^2 + y^2}{c^2\,(1 + \lambda_2^2)} + \frac{z^2}{c^2 \lambda_2^2} = 1.$$

If the initial conditions are so specified that <u>both</u> roots, λ_1 and λ_2, are less than -30, then satellite motions will occur. If, on the other hand, $\lambda_2 > -30$, ballistic trajectories will occur.

The formulas for the coordinates μ, λ, w are of the form

$$\left.\begin{aligned}
\mu &= \mu_2 \operatorname{sn} \sigma (\tau - \tau_0), \\
\lambda &= \frac{A + B \cos u}{C + D \cos u}, \\
w &= c_5 - \frac{c_1}{1 + \lambda_3^2} \tau + \frac{c_1}{\sigma} \prod (\varphi, n, k) + \\
&+ \frac{2c_1}{\sqrt{2h (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}} \left\{ \frac{M_1}{2} \ln \left[\left(\tan \frac{u}{2} - m_1\right)^2 + \right.\right. \\
&\left. + n_1^2 \right] + \\
&+ \frac{M_2}{2} \ln \left[\left(\tan \frac{u}{2} - m_2\right)^2 + n_2^2 \right] + \frac{M_1 m_1 + N_1}{n_1} \arctan \frac{\tan \frac{u}{2} - m_1}{n_1} + \\
&\left. + \frac{M_2 m_2 + N_2}{n_2} \arctan \frac{\tan \frac{u}{2} - m_2}{n_2} \right\},
\end{aligned}\right\} \tag{98}$$

where /191

$$u = \sqrt{2h\,(\lambda_3 - \lambda_1)\,(\lambda_3 - \lambda_2)}\,(\tau - \tau_1)\ , \quad (99)$$

$$\left.\begin{aligned} A &= \lambda_3\,(\lambda_1 + \lambda_2) - 2\lambda_1\lambda_2, \quad B = \lambda_3\,(\lambda_2 - \lambda_1), \\ C &= 2\lambda_3 - \lambda_1 - \lambda_2, \quad D = \lambda_2 - \lambda_1. \end{aligned}\right\} \quad (100)$$

The coefficients M_1, M_2, N_1, N_2 are defined in terms of the following system of algebraic equations:

$$\left.\begin{aligned} & M_1 + M_2 = 0, \\ & -2m_2M_1 - 2m_1M_2 + N_1 + N_2 = \overline{A}, \\ & M_1(m_2^2 + n_2^2) + M_2(m_1^2 + n_1^2) - 2m_2N_1 - 2m_1N_2 = 0, \\ & N_1\,(m_2^2 + n_2^2) + N_2\,(m_1^2 + n_1^2) = \overline{B}, \end{aligned}\right\} \quad (101)$$

$$\left.\begin{aligned}\bar{A} &= \frac{B^2C\,(C - 2D) - A^2D\,(D - 2B)}{B^2 + D^2}\,,\\[2ex] \bar{B} &= \frac{B^2C\,(C + 2D) - A^2D\,(D + 2B)}{B^2 + D^2}\,,\end{aligned}\right\} \tag{102}$$

$$\left.\begin{aligned}m_1 &= -\,m_2 = \sqrt{\frac{(\lambda_1 - \lambda_3)\,\left(\sqrt{1 + \lambda_1^2 + \lambda_2^2 + \lambda_2^2\lambda_2^2} - 1\right)}{2\,(\lambda_2 - \lambda_3)\,(1 + \lambda_1^2)}}\\[2ex] n_1 &= n_2 = \sqrt{\frac{(\lambda_3 - \lambda_1)\,(\lambda_2 - \lambda_1)\,\left(\sqrt{1 + \lambda_1^2 + \lambda_2^2 + \lambda_1^2\lambda_2^2} + 1\right)}{2(\lambda_3 - \lambda_2)\,(1 + \lambda_1^2)\,(\lambda_1^2 + \lambda_2^2 + \lambda_1^2\lambda_2^2)}}\,.\end{aligned}\right\} \tag{103}$$

The time t depends on the variable τ in accord with the formula

$$t = c_6 + (\mu_1^2 + \lambda_1^2)\,\tau - \frac{\mu_1^2}{\sigma}\,E\,(\varphi,\ k) +$$

$$+ \frac{1}{4\sqrt{2h}(\lambda_3 - \lambda_1)^{3/2}(\lambda_3 - \lambda_2)^{3/2}} \left\{ \frac{A_2}{C + D \cos u} + B_2 \operatorname{arc\,tan} \left(s_1 \tan \frac{u}{2} \right) \right\} \tag{104}$$

where

$$\left. \begin{aligned} A_2 &= -\frac{(AD - BC)^2}{D}, \qquad s_1 = \sqrt{\frac{C - D}{C + D}}, \\ B_2 &= \frac{2}{\sqrt{C^2 - D^2}} \left[\frac{C(A^2D^2 - B^2C^2)}{D^2} - 2\,(ABD - B^2C) \right]. \end{aligned} \right\} \tag{105}$$

The regions of space in which the motion of the point mass will take place are depicted in Fig. 5.

11. THE CASE j ($\lambda_1 = \lambda_2$; ROOTS λ_3 and λ_4 IMAGINARY)

The polynomial $\varphi(\lambda)$ for this case is plotted in Fig. 12.

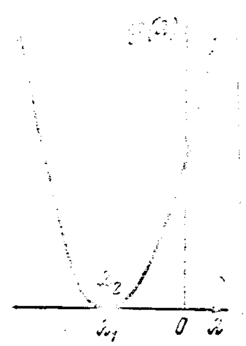

Fig. 12. Slope of polynomial $\varphi(\lambda)$ in the case $\lambda_1 = \lambda_2$ and imaginary λ_3 and λ_4.

We now prove that when the polynomial $\varphi(\lambda)$ has two imaginary roots and one real root of multiplicity 2, real motions are possible on the ellipsoid $\lambda = \lambda_1$. For this case, the relationship linking the roots and the coefficients of $\varphi(\lambda)$ are represented by the equations

$$\left.\begin{aligned} &2\lambda_1 + 2p = -a, \\ &\lambda_1^2 + 4\lambda_1 p + (p^2 + q^2) = b \\ &2\lambda_1^2 p + 2\lambda_1 (p^2 + q^2) = -a, \\ &\lambda_1^2 (p^2 + q^2) = d, \end{aligned}\right\} \tag{106}$$

and $\lambda_3 = p + iq$, $\lambda_4 = p - iq$.

We shall now show that the equations (106) and the inequality (28) are valid at values $\lambda_1 < -30$. From the equations (106), we may derive /192

$$p = \frac{\lambda_1}{1 - \lambda_1^2} (p^2 + q^2 - 1), \tag{107}$$

$$(1 - \lambda_1^2)\ (p^2 + q^2 - 1) > -4\lambda_1 p. \tag{108}$$

Inequality (108), as the counterpart inequalities above, expresses the constraint $b > d + 1$. Inequality (108) may be stated as

$$\frac{(1 - \lambda_1^2)\ (p^2 + q^2 - 1)^2}{\lambda_1\ (p^2 + q^2 - 1)} < - 4p. \qquad (109)$$

At $\lambda_1 < - 30$ the fraction $\frac{1 - \lambda_1^2}{\lambda_1}$ is positive, so that the following inequality holds:

$$\frac{(p^2 + q^2 - 1)^2}{p} < - 4\ p. \qquad (110)$$

Inequality (110) holds when $p < 0$. The quantity $p^2 + q^2$ may be determined from the formula (107):

$$p^2 + q^2 - 1 = \frac{p(1 - \lambda_1^2)}{\lambda_1} \qquad (111)$$

Assuming $\lambda_1 < - 30$, and taking $p < 0$ into account, we shall have the constraints:

$$0 < p^2 + q^2 < 1. \qquad (112)$$

for $p^2 + q^2$. Accordingly, $-1 < p < 0$. At these values of p, inequality (108) will always hold.

We see then that real motions are possible and that they will occur on the ellipsoid

$$\frac{x^2 + y^2}{c^2(1 + \lambda_1^2)} + \frac{z^2}{c^2\lambda_1^2} = 1.$$

The formulas for the elliptic coordinates μ, λ, w and the rectangular coordinates x, y, z are listed in Section 7 (formula (73)].

12. THE CASE k ($\lambda_1 < \lambda_2 < 0$; ROOTS λ_3 and λ_4 IMAGINARY)

The behavior of the polynomial $\varphi(\lambda)$ is graphed in Fig. 13.

We shall prove that when the polynomial $\varphi(\lambda)$ has two real and two imaginary roots, then both satellite and ballistic trajectories will occur among the real motions. The roots and coefficients of $\varphi(\lambda)$ are interrelated by the formulas

$$\left.\begin{aligned} &\lambda_1 + \lambda_2 + 2p = -a, \\ &\lambda_1\lambda_2 + 2p(\lambda_1 + \lambda_2) + (p^2 + q^2) = b, \\ &2p\lambda_1\lambda_2 + (\lambda_1 + \lambda_2)(p^2 + q^2) = -a, \\ &\lambda_1\lambda_2(p^2 + q^2) = d. \end{aligned}\right\} \quad (113)$$

Bearing in mind $b > d + 1$ and the fact that the right-hand members of the first and third equations in (113) are equal to $- a$, we may arrive at

$$\left.\begin{aligned} &2p\ (\lambda_1\lambda_2 - 1) + (\lambda_1 + \lambda_2)\ (p^2 + q^2 - 1) = 0, \\ &p[(\lambda_1\lambda_2 - 1)^2 + (\lambda_1 + \lambda_2)^2] < 0. \end{aligned}\right\} \quad (114)$$ /193

We see from inequality (114) that

$$p < 0. \quad (115)$$

For satellite trajectories, $\lambda_1 < - 30$ and $\lambda_2 < - 30$, so that $\lambda_1\lambda_2 - 1 > 0$, and $\lambda_1 + \lambda_2 < 0$. But in that case, (114) may occur only when

$$0 < p^2 + q^2 < 1. \quad (116)$$

For ballistic trajectories, the constraint (116) is not mandatory.

The formulas for the elliptic coordinates μ and λ have the form

$$\left.\begin{aligned} \mu &= \mu_2 \,\mathrm{sn}\, \sigma \;(\tau - \tau_0), \\ \lambda &= \frac{A + B \,\mathrm{cn}\, \sigma_1 \;(\tau - \tau_1)}{C + D \,\mathrm{cn}\, \sigma_1 \;(\tau - \tau_1)}\,, \end{aligned}\right\} \tag{117}$$

where

$$\left.\begin{aligned} A &= -\, m\lambda_1 - n\lambda_2, \qquad & B &= -\, m\lambda_1 + n\lambda_2, \\ C &= -\, m - n, & D &= -\, m + n, \end{aligned}\right\} \tag{118}$$

$$\left.\begin{aligned} m &= \sqrt{(p - \lambda_2)^2 + q^2}\,, \\ n &= \sqrt{(p - \lambda_1)^2 + q^2}\,, \end{aligned}\right\} \tag{119}$$

$$\sigma_1 = \sqrt{2hmn}. \tag{120}$$

The modulus of the function cn σ_1 $(\tau - \tau_1)$ is expressed by the formula

$$k_1 = \frac{1}{2}\sqrt{\frac{(\lambda_2 - \lambda_1)^2 - (m - n)^2}{mn}} \quad . \tag{121}$$

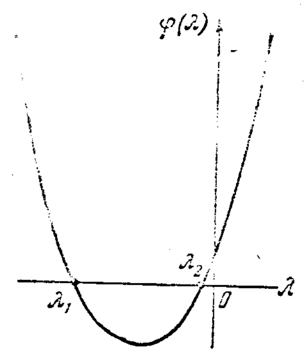

Fig. 13. Slope of polynomial $\varphi(\lambda)$ in the case $\lambda_1 < \lambda_2 < 0$ and λ_3, λ_4 imaginary.

Inequality (116) demonstrates that, for satellite motions, the modulus k_1 is an extremely small quantity.

The coordinate w is determined from the formula

$$w = c_5 + \frac{c_1}{\sigma}\prod(\varphi, n, k) - c_1 J_2. \tag{122}$$

The integral J_2, just as in case d, is represented as the sum of two elliptic integrals of the third kind with complex conjugate parameters. Using the Hoüel transformation [4], we may restate the integral J_2 in the form

$$J_2 = \frac{D^2}{B^2 + D^2}\tau + \iota_1 L_1 + \iota_2 L_2 ,$$

where the functions L_1 and L_2 are of the form specified by (60). The relationship between t and τ is given by a formula of the form (62).

As was mentioned in Section 5, formulas (61) and (62) are hardly suited to use in computations of concrete satellite orbits. It would be much quicker to obtain numerical results having the required accuracy by a series expansion of the integrand functions in powers of small values of k and k_1.

The formulas for the rectangular coordinates x, y, and z are of the form /194

$$\left.\begin{aligned} x &= c\,\frac{\sqrt{1-\mu_2^2\,\mathrm{sn}^2\,u}\,\sqrt{(A+B\,\mathrm{cn}\,\upsilon)^2+(C+D\,\mathrm{cn}\,\upsilon)^2}}{C+D\,\mathrm{cn}\,\upsilon}\sin w,\\ y &= c\,\frac{\sqrt{1-\mu_2^2\,\mathrm{sn}^2\,u}\,\sqrt{(A+B\,\mathrm{cn}\,\upsilon)^2+(C+D\,\mathrm{cn}\,\upsilon)^2}}{C+D\,\mathrm{cn}\,\upsilon}\cos w,\\ z &= -c\mu_2\,\mathrm{sn}\,u\,\frac{A+B\,\mathrm{cn}\,\upsilon}{C+D\,\mathrm{cn}\,\upsilon}, \end{aligned}\right\} \tag{123}$$

$$\left.\begin{aligned} u &= \sigma\,(\tau - \tau_0), \\ \upsilon &= \sigma_1\,(\tau - \tau_1). \end{aligned}\right\} \tag{124}$$

The trajectory lies between two confocal ellipsoids $\lambda = \lambda_1$ and $\lambda = \lambda_2$, and outside the hyperboloid

$$\frac{x^2 + y^2}{c^2\,(1 - \mu_2^2)} - \frac{z^2}{c^2\mu_2^2} = 1.$$

The regions of space in which the motion occurs are shown in Fig. 5.

13. SOME CONCLUSIONS

The analysis carried out in the preceding sections makes it possible to arrive at certain conclusions. If an arbitrary integration constant h is positive, then the motion of a point mass will either occor between two confocal ellipsoids of low eccentricity ($e_1 = \frac{1}{\sqrt{1 + \lambda_1^2}}$, $e_2 = \frac{1}{\sqrt{1 + \lambda_2^2}}$) , or on the ellipsoid itself. All motions of the elliptical class will occur in a restricted portion of space. If the ellipsoid of smaller size is located in the interior of the earth, then the motion of the point will take place between the earth's surface and the larger ellipsoid.

The rectangular coordinates x, y, and z are expressed, in the principal cases where the polynomial $\varphi(\lambda)$ does not have multiple roots, in terms of periodic functions of unequal periods. The elliptic coordinate μ has a real period

$$T = 4K(k), \quad K(k) = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} . \tag{125}$$

The real period of the variable λ is

$$T_1 = 4K(k_1), \quad K(k_1) = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - k_1^2 \sin^2 x}} . \tag{126}$$

The functions sin w and cos w are of period 2π. In general, x, y, and z are not periodic functions of the regularizing time τ, and the motion of the point will therefore occur on nonclosed space curves. If the periods T, T_1, and 2π are commensurate for some of the initial data, then the motion of the point will in that case be periodic with respect to τ, although this does not imply that it will be periodic with respect to the time t. /195

Case d and k may be termed principal cases, since in these cases the polynomial $\varphi(\lambda)$ does not have multiple roots. The case i is transitional between cases d and k. For each set of initial data, there exists some limiting inclination: at inclinations less than the limiting value, the polynomial $\varphi(\lambda)$ has four real and unequal roots. If the inclination is greater than that indicated, then the polynomial $\varphi(\lambda)$ will have two real and two imaginary roots.

We here introduce several concepts to afford a more convenient description of the motion. The draconic period of revolution of a satellite on the j-th circuit will be a term for the quantity defined by the formula

$$T^{j}\ \Omega = \int_{\tau^{j-1}}^{\tau^{j}} (\mu^2 + \lambda^2)\ d\tau\ , \tag{127}$$

where τ^{j-1} and τ^{j} are solutions of the equation $z = 0$ or, in other words, of the equation

$$\mathrm{sh}\ [\sigma\ (\tau - \tau_0)] = 0. \tag{128}$$

The quantities τ^{j-1} and τ^{j} used here are two successive instants of the regularizing time variable τ, at which the satellite passes from the southern hemisphere into the northern. The real solutions of equation (128) are found from the formula

$$\tau^{j} = \tau_0 + \frac{4j}{\sigma} \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}\ . \tag{129}$$

For each of the cases treated here, we obtain a specified expression for the draconic period of revolution. The simplest formula is obtained for motion on an ellipsoid

$$T_{\Omega}^{(s)} = (\mu_1^2 + \lambda_1^2)\,(\tau^s - \tau^{s-1}) -$$

$$- \frac{\mu_1^2}{\sigma}\,[E(\mathrm{am}[\sigma(\tau^s - \tau_0); \;\; k) - E(\mathrm{am}[\sigma(\tau^{s-1} - \tau_0)]; \; k)]. \tag{130}$$

We use the term quasiperigee of a trajectory to describe the point at which the trajectory is tangent to the smaller ellipsoid, and the term quasiapogee to describe the point of tangency with the larger ellipsoid. The quasiperigee and quasiapogee are, generally speaking, not points at which the geocentric distance of the satellite assumes extreme values on a given circuit. The term quasianomalistic period of revolution of a satellite on the i-th circuit will be applied to the time interval between two successive passages of a point through the quasiperigee. The quasianomalistic period is computed using the formula

$$T_m^{(i)} = \int_{\tau_m^{i-1}}^{\tau_m^i} (\mu^2 + \lambda^2)\,d\tau, \tag{131}$$

where τ_m^{i-1} and τ_m^i are solutions of the equation

$$\frac{A + B\;\mathrm{sn}^2\,\sigma_1\,(\tau - \tau_1)}{C + D\;\mathrm{sn}^2\,\sigma_1\,(\tau - \tau_1)} = \lambda_2 \tag{132}$$

for the case d, and of the equation

$$\frac{A + B \operatorname{cn} \sigma_1 (\tau - \tau_1)}{C + D \operatorname{cn} \sigma_1 (\tau - \tau_1)} = \lambda_2 \tag{133}$$

for the case k.

The change in the longitude of the ascending node on the j-th circuit is computed using the formula /196

$$\Omega'_j = w(\tau^{j-1}) - w(\tau^j). \tag{134}$$

In each case, the coordinate w is computed by means of the appropriate parametric formula.

For the change in the angular distance of the quasiperigee from the node on the i-th circuit of the satellite (the angle between the radius vectors of two adjacent quasiperigees), we may readily derive the formula

$$\cos w'_i = \frac{\sqrt{[1+(\lambda^i)^2][1+(\lambda^{i-1})^2][1-(\mu^i)^2][1-(\mu^{i-1})^2]}\cos(w^i-w^{i-1})}{r^{i-1}r^i} + \frac{\lambda^{i-1}\lambda^i\mu^{i-1}\mu^i}{r^{i-1}r^i}, \tag{135}$$

where λ^{i-1}, λ^{i}, μ^{i-1}, μ^{i}, w^{i-1}, w^{i}, r^{i-1}, r^{i} are the values of the elliptic coordinates and of the radius vector for the times τ_m^{i-1} and τ_m^{i}.

In concrete computational work, the results mentioned in the article may be put to use as follows. Using the specified initial data (initial position x_0, y_0, z_0 and initial velocity $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$), we compute the arbitrary constants h, c_1, and c_2. The relationship between the arbitrary constants and the initial data is found by recourse to the formulas

$$c_1 = \dot{w}_0 \, (1 - \mu_0^2) \, (1 + \lambda_0^2), \qquad (136)$$

$$h = - \frac{fM\lambda_0}{c^3(\mu_0^2 + \lambda_0^2)} - \frac{\dot{w}_0^2(1 - \mu_0^2)(1 + \lambda_0^2)}{2} - \frac{\mu_0^2 + \lambda_0^2}{2}\left[\frac{\dot{\mu}_0^2}{1 - \mu_0^2} + \frac{\dot{\lambda}_0^2}{1 + \mu_0^2}\right], \qquad (137)$$

$$c_2 = - h\mu_0^2 - \frac{\dot{\mu}_0^2}{(21 - \mu_0^2)} - \frac{c_1^2}{1 - \mu_0^2} \;. \qquad (138)$$

The initial values of the elliptic coordinates μ_0, λ_0, w_0 and their time derivatives $\dot{\mu}_0$, $\dot{\lambda}_0$, and $\dot{w}_0$ are computed using the formulas

$$c^2\mu_0^4 + \mu_0^2 (r_0^2 - c^2) - z_0^2 = 0, \tag{139}$$

$$c^2\lambda_0^4 - \lambda_0^2 (r_0^2 - c^2) - z_0^2 = 0, \tag{140}$$

$$\tan w_0 = \frac{x_0}{y_0}, \tag{141}$$

$$\dot{\mu}_0 \left[\frac{(x_0^2 + y_0^2)\,\mu_0}{(1 - \mu_0^2)^2} - \frac{z_0^2}{\mu_0^3}\right] = \frac{x_0\dot{x}_0 + y_0\dot{y}_0}{1 - \mu_0^2} - \frac{z_0\dot{z}_0}{\mu_0^2}, \tag{142}$$

$$\dot{\lambda}_0 \left[\frac{(x_0^2 + y_0^2)\lambda_0}{(1 + \lambda_0^2)^2} - \frac{z_0^2}{\lambda_0^3}\right] = \frac{x_0\dot{x}_0 + y_0\dot{y}_0}{1 + \lambda_0^2} - \frac{z_0\dot{z}_0}{\lambda_0^2}, \tag{143}$$

$$\dot{w}_0 = \frac{\sin w_0 \cos w_0 (\dot{x}_0 \cos w_0 - \dot{y}_0 \sin w_0)}{x_0 \cos^3 w_0 + y_0 \sin^3 w_0}. \tag{144}$$

The determination of λ_0, μ_0, and w_0 is unique, since $\lambda_0 < 0$, /197
$0 < \mu_0 < 1$, and w_0 is chosen with the signs of x_0 and y_0

taken into account.

If the arbitrary constants h, c_1, and c_2 are known, we then proceed to compute the roots of the polynomials $f(\mu)$ and $\varphi(\lambda)$. Finding the roots of the polynomial $\varphi(\lambda)$ is made easier by the fact that the approximate values of two roots are known from the initial data, in the case of satellite motions: one root is approximately equal to the ratio of the perigee distance to the value of c, while the other root is approximately equal to the ratio of the apogee distance to c. In the case of ballistic trajectories, one of the roots is approximately equal to the ratio of the distance from the center to the furthest removed point on the trajectory to the value of c. After the roots of the polynomials $f(\mu)$ and $\varphi(\lambda)$ have been determined, the case to which the given motion corresponds will be known. Moreover, in investigating the motion, it becomes necessary to rely on formulas for elliptic and rectangular coordinates corresponding to the specific case.

The use of the formulas derived here to describe the motion of a concrete satellite does not prove to be convenient in every instance. The occasional inconvenience is due to the fact that detailed tabular data with a large number of places for elliptic integrals of the first and second kinds, and especially for elliptic integrals of the third kind, are lacking in the literature. These problems can be avoided with ease by resorting to series expansions of the solutions obtained in powers of small quantities of the order of the flattening of the earth. The series expansion of the solution will be published in a separate article.

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